Q.1. a) $(3-\lambda)(-1-\lambda)+4=0$ —— (1)

$-(3-\lambda)(1+\lambda)+4=0$

$-(3-\lambda)+\lambda(-3+\lambda)+4=0$

$-3+\lambda-3\lambda+\lambda^2+4=0$

$\lambda^2-2\lambda+1=0 \quad \lambda=1,1.$ —— (2)

Let $A^{100}=aA+b.$ —— (3)

$\lambda^{100}=a\lambda+b \quad \therefore \quad 100\lambda^{99}=a$

$1=a+b$ —— (A) $\quad \boxed{100=a} \quad \therefore \quad \underline{b=-99}$ —— (4)

$A^{100}=100A-99I=\begin{pmatrix}300 & -400\\ 100 & -100\end{pmatrix}-\begin{pmatrix}99 & 0\\ 0 & 99\end{pmatrix}=\begin{pmatrix}201 & -400\\ 100 & -199\end{pmatrix}$ —— (5)

Q.1. b) $u_x=v_y$ —— (1) | $u_y=-v_x$ —— (2)

$2x+2ay=2dx+2y$ | $2ax+2by=-(2cx+2dy)$

$d=1, \quad a=1$ —— (3) | $a=-c, \quad b=-d$

$c=-1, \quad b=-1$ —— (4)

Ans: $a=1, \quad b=-1, \quad c=-1, \quad d=1.$ —— (5)

Q.1. c) $I=\int_0^{1+i} z^2\,dz=\int_0^{1+i}(x+iy)^2(dx+i\,dy)$ —— (1) For $y=x$, $dy=dx$

$=\int_0^1(x+ix)^2(dx+i\,dx)$ —— (2) $=\int_0^1(1+i)^3x^2\,dx=$ —— (3)

$=(1+i)^3\left[\frac{x^3}{3}\right]_0^1=\frac{(1+i)^3}{3}=$ —— (4)

$=\frac{1+3i+3i^2-i}{3}=\frac{1+3i-3-i}{3}=\frac{-2+2i}{3}=\underline{-\frac{2}{3}+\frac{2}{3}i}$ —— (5)

Q.1. D)

1	2	3	4	5	
P	q	$\sim P$	$q \wedge P$	$(q \wedge \sim P)$	$(4) \vee (5)$
T	T	F	T	F	T
T	F	F	F	F	F
F	T	T	F	T	T
F	F	T	F	F	F —— (3)

The Final column contains T & F. —— (4)

$\Rightarrow$ It is contingency. —— (5)

Q.1. E)

i) Assume $f(x_1) = f(x_2)$

$$\frac{4x_1+3}{5x_1-2} = \frac{4x_2+3}{5x_2-2}$$

$\therefore (4x_1+3)(5x_2-2) = (4x_2+3)(5x_1-2)$

$\therefore 20x_1x_2 - 8x_1 + 15x_2 - 6 = 20x_1x_2 - 8x_2 + 15x_1 - 6$

$-8x_1 + 15x_2 = -8x_2 + 15x_1$.

$\therefore -23x_1 = -23x_2 \Rightarrow \underline{x_1 = x_2}$

Thus, f is injective.

ii) Let $y = f(x)$

$\therefore y = \frac{4x+3}{5x-2}$

$\therefore 5xy - 2y = 4x + 3$

$(5y-4)x = 3+2y$

$$x = \frac{3+2y}{5y-4} = \frac{2y+3}{5y-4}, \quad y \neq \frac{4}{5}.$$

$\Rightarrow f$ is surjective.

$\therefore$ since f is injective & surjective.

$\Rightarrow f^{-1}(x) = \frac{2x+3}{5x-4}$.

Q.1 (F) Number of students (objects) = 25

" months (pigeonholes) = 12 ——— (1)

By extended pigeonhole principle, $\left\lceil \frac{25}{12} \right\rceil = \lceil 2.083 \rceil = 3$ ——— (2)

$\therefore$ at least 3 students must have their birthdays in the same month. ——— (3)

At least 3 students ~~have~~ share the same birth month. ——— (5)

Q.2. A) Trace = 2, SMD = $4-12-3-(-12)$, $|A| = 1(0)+0+0 = 0$ (2)

$= 4-12-3+12$

$= 1.$

$\lambda^3 - 2\lambda^2 + \lambda = 0$

$\lambda(\lambda^2 - 2\lambda + 1) = 0 \Rightarrow \lambda = 0, 1, 1,$ ——— 3

For $\lambda = 0$, $X_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, ——— 6

For $\lambda = 1$, $\begin{pmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-6x_2 - 4x_3 = 0$

$3x_2 + 2x_3 = 0$

$x_2 = -\frac{2}{3}x_3$

$X_2 = (1, -2, 3)$ or $(1, 0, 0)$.

$X_3 = (0, -2, 3)$

$r = 1, n = 3$

$n - r = 3 - 1 = 2$

put $x_1 = k_1 = 1$, $x_3 = 3$

Any values of x_1, & x_3

——— 8

A.M = G.M for all λ.

A is diagonalisable.

$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & -2 \\ 2 & 0 & 3 \end{pmatrix}$ ——— 10

Q.2. B) Trace = 7

SMD $= -9 - 21 + 21 - (-20 + 15 - 20)$

$= -9 + 20 - 15 + 20 = 40 - 24 = 16.$

$|A| = 12.$

$\lambda^3 - 7\lambda^2 - 16\lambda - 12 = 0.$

To prove, $A^3 - 7A^2 + 16A - 12I = 0.$

$A^2 = \begin{pmatrix} 4 & 25 & 10 \\ -12 & -31 & -26 \\ 20 & 50 & 44 \end{pmatrix}$, $A^3 = \begin{pmatrix} -8 & -25 & -10 \\ -12 & -31 & -26 \\ 20 & 50 & 44 \end{pmatrix}$.

To find A^{-1}, $A^2 - 7A - 16I - 12A^{-1} = 0$

$A^{-1} = \frac{1}{12}(A^2 - 7A - 16I) =$

$A^{-1} = \begin{pmatrix} \frac{-1}{12} & \frac{-15}{4} & \frac{-25}{12} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \\ \frac{-1}{12} & \frac{5}{4} & \frac{11}{12} \end{pmatrix}$

Q3. A).

$A^T = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$

$A^TA = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ $\therefore |A^TA - \lambda I| = 0.$

$\begin{vmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$

$(2-\lambda)(5-\lambda) - 9 = 0.$

$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$, $A^T = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$ — 1

$A^TA = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$ $\therefore |A^TA - \lambda I| = 0$

$\Rightarrow \lambda = 16, 1.$ — 2

$\lambda_1 = 16, \quad \lambda_2 = 1.$

$\sigma_1 = \sqrt{16} = 4, \quad \sigma_2 = 1.$ $D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ — 3

For $\lambda = 16$, $\begin{pmatrix} 4-16 & 6 \\ 6 & 13-16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore +2x_1 - x_2 = 0 \quad \therefore 2x_1 = x_2$

$X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ — (5)

For $\lambda = 1$, $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\therefore 3x_1 + 6x_2 = 0$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$X_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ — (6)

Since, $V_1 \cdot V_2 = 0$. V_1 & V_2 are orthogonal.

$v_1 = \frac{V_1}{\|V_1\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. $v_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ — (7)

$u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. $u_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ — (8)

$\therefore U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$ — (9)

$\therefore \underline{A = UDV^T}$ — (10)

Q.3 . B. Theory . - Definition - 2M.

Mathematical Representation - $A = U \Sigma V^T$.

Applications → ① Digital Image compression.

② PCA & Dimensionality Reduction.

③ Recommender systems.

④ Signal processing & Noise Reduction.

Q.4 . A) Statement: $a^{p-1} \equiv 1 \pmod{p}$, —(2M)

p = prime, a = Integer.

i) $a = 3$, $p = 17$, $a \times p \Rightarrow 3 \times 17$.

$a^{p-1} \equiv 1 \pmod{p}$ —(3M)

$3^{16} \equiv 1 \pmod{17}$ —①

—(4M)

Remainder = 1.

ii) $a = 2$, $p = 11$, —(5M)

$2^{10} \equiv 1 \pmod{11}$. —(6M)

$103 = (10 \times 10) + 3$.

$2^{103} = (2^{10})^{10} \times 2^3$

$2^{103} \equiv (1)^{10} \times 2^3 \pmod{11}$ —(8M)

$2^{103} \equiv 8 \pmod{11}$

—(10M)

∴ Remainder = 8

Q.4.(B) Euler's Theorem:

If $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$ —2

where $\phi(n)$ = Euler's totient function. —3

$a = 6$, $n = 25$, $\gcd(6, 25) = 1$.

$\phi(25) = \phi(5^2) = 25 \times (1 - \frac{1}{5}) = 25 \times \frac{4}{5} = 20$ —④

∴ $6^{20} \equiv 1 \pmod{25}$ —①

we have, $6 \equiv 6 \pmod{25}$

6^5 —⑥

—⑤

$6^2 = 36 \equiv 11 \pmod{25}$ — (6)

$6^3 = 216 \equiv 16 \pmod{25}$

$6^4 = 96 \equiv 21 \pmod{25}$ — (7)

$6^5 = 126 \equiv 1 \pmod{25}$ — (8)

$99 = 5 \times 19 + 4$

$\therefore 6^{99} \equiv (6^5)^{19} \times 6^4 \equiv (1)^{19} \times 6^4 \pmod{} = (1)^{19} \times 21 \equiv 21 \pmod{25}$ — (10)

∴ The remainder is 21.

Q.5. A) $v = 3x^2y + 6xy - y^3$

$\psi_1 = v_x = 6xy + 6y$ ∴ $v_{xx} = 6y$

$\psi_2 = v_y = 3x^2 + 6x - 3y^2$ $v_{yy} = -6y$ ∴ $v_{xx} + v_{yy} = 0$ — (3)

$f'(z) = \int [\psi_2(z,0) + i\,\psi_1(z,0)]\,dz + c$ — (4)

$= \int [3z^2 + 6z + i(0)]\,dz + c$ — (5)

$= 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + c$ — (6)

$= z^3 + 3z^2 + c$ put $z = x+iy$ — (7)

$= (x+iy)^3 + 3(x+iy)^2 + c$

$= x^3 + 3x^2 iy + 3x i^2 y^2 + i^3 y^3 + 3(x^2 + 2ixy + i^2 y^2) + c$

∴ $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + c_1$ → Harmonic conjugate. — (10)

Q.5. B) Let $u = 2x - x^3 + 3xy^2 = k$.

∴ $u_x = 2 - 3x^2 + 3y^2 = \phi_1$ — (1)

$u_y = 6xy = \phi_2$ — (2)

$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz$ — (4)

$= \int [(2 - 3z^2) - i(0)]\,dz + c$ $= 2z - 3 \cdot \frac{z^3}{3} + c$ — (5)

$= 2(x+iy) - (x+iy)^3 + c$ — (6)

$= 2x + 2iy - (x^3 + 3x^2 iy - 3xy^2 - iy^3) + c$ — (8)

$v = (2y - 3x^2 y + y^3) + c_1$ → orthogonal projectory. — (10)

Q.6. A) $\oint_C f(z)\,dz = 2\pi i$ (sum of residues at its poles) — (2M) (4)

$4z^2+1=0 \quad \therefore z^2 = -\frac{1}{4} \quad \therefore z = \pm\frac{i}{2}$ — (3M)

$|z| = \left|\frac{i}{2}\right| = \sqrt{\frac{1}{4}} = \frac{1}{2} = 0.5 < 1.$

$|z| = \left|\frac{-i}{2}\right| = \sqrt{\frac{1}{4}} = 0.5 < 1.$ — (4M)

Both poles lies inside C : $|z|=1$

$\text{Res}\left(f, i/2\right) = \lim_{z\to\frac{i}{2}} \left(z-\frac{i}{2}\right)\cdot\frac{1}{4\left(z-\frac{i}{2}\right)\left(z+\frac{i}{2}\right)}$

$= \frac{1}{4}\cdot\frac{1}{i} = -\frac{i}{4}$ — (6M)

$\text{Res}\left(f, -i/2\right) = \lim_{z\to-\frac{i}{2}} \left(z+\frac{i}{2}\right)\cdot\frac{1}{4\left(z-\frac{i}{2}\right)\left(z+\frac{i}{2}\right)}$

$= \frac{1}{4}\cdot\frac{1}{(-i)} = \frac{i}{4}$ — (8M)

$\therefore \oint \frac{dz}{4z^2+1} = 2\pi i\left(-\frac{i}{4}+\frac{i}{4}\right) = 0.$ — (10M)

Q.6. B) $f(z) = \frac{z}{(z-1)(z-2)}$ Let $\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$z = A(z-2) + B(z-1)$

pwt $z=1$

$1 = A(1-2) \therefore A = -1$

pwt $z=2$

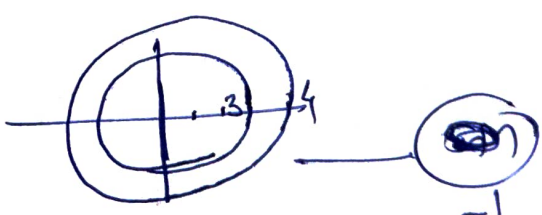
$2 = B(2-1) \therefore B = 2$

$f(z) = \frac{-1}{z-1} + \frac{2}{z-2}$ — (2M)

$f(z) = \frac{-1}{(z+2)-3} + \frac{2}{(z+2)-4}$ — (3M)

$f(z) = \frac{-1}{u-3} + \frac{2}{u-4}$

$= \frac{1}{3-u} + \frac{2}{u-4}$ — (4M)



— (5M)

R_1: $|u|<3$, R_2: $3<|u|<4$, R_3: $|u|>4$

$f(z) = \frac{+1}{3\left(1-\frac{u}{3}\right)} + \frac{2}{4\left(1-\frac{u}{4}\right)} = \frac{1}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{1}{2}\left(1-\frac{u}{4}\right)^{-1}$

$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{u}{3}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{u}{4}\right)^n$ — (6M)

$$f(z) = \frac{1}{3-u} + \frac{2}{u-4}$$

ii) R_2: $3 < |u| < 4$

$$\left|\frac{3}{u}\right| < 1, \quad \left|\frac{u}{4}\right| < 3$$

$$f(z) = \frac{1}{u\left(\frac{3}{u}-1\right)} - \frac{2}{4\left(1-\frac{u}{4}\right)}$$

$$= -\frac{1}{u}\left(1-\frac{3}{u}\right)^{-1} - \frac{1}{2}\left(1-\frac{u}{4}\right)^{-1}$$

$$= -\frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{3}{u}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{u}{4}\right)^n \quad \text{—— (8M)}$$

iii) R_3: $|u| > 4$, $\Rightarrow |u| > 3$

$$\left|\frac{4}{u}\right| < 1, \quad \left|\frac{3}{u}\right| < 1$$

$$f(z) = \frac{1}{u\left(\frac{3}{u}-1\right)} + \frac{2}{u\left(1-\frac{4}{u}\right)}$$

$$= -\frac{1}{u}\left(1-\frac{3}{u}\right)^{-1} + \frac{2}{u}\left(1-\frac{4}{u}\right)^{-1}$$

$$f(z) = -\frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{3}{u}\right)^n + \frac{2}{u}\sum_{n=0}^{\infty}\left(\frac{4}{u}\right)^n \quad \text{—— (10M)}$$